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Kinematics of Dirac's spinor field

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Abstract. A kinematical separation of the slowly varying parts from the violently oscillating constituents of Dirac's spinor field is achieved, which also carries over to the physical densities, e.g. to the current and the polarization. Various applications of this separation are discussed, such as the Gordon decomposition and the exclusion of a *closed* universe by the coupled Dirac–Einstein equations.

1. Introduction

Retrospectively, the historical breakthrough of quantum mechanics was already achieved by the twenties of our century, when the pioneers [1] of the new quantum paradigm became able to explain the spectral lines of atoms and molecules. In particular the Bohr–Heisenberg–Schrödinger debate about the correct explanation of the spectral frequencies brought out clearly the strange and even paradoxical features of the quantum approach: it is not the orbital period of the electron in an hydrogen atom which determines the frequency of the emitted light (as Schrödinger originally believed), but it is the energy difference $\Delta E = E_{\mathrm{II}} - E_{\mathrm{I}}$ of the two stationary states involved, divided by Planck's constant $\hbar$, which leads to the correct relationship

$$\omega = \frac{\Delta E}{\hbar} \tag{1.1}$$

nowadays known as 'Bohr's frequency condition'.

As mysterious this result may appear from the classical point of view, it is correspondingly evident through a quantum-mechanical (or better, semi-classical) argument: when the electron is in the upper eigenstate $|\psi_{\mathrm{II}}\rangle$ of the Hamiltonian

$$\hat{H}|\psi_{\mathrm{II}}\rangle = E_{\mathrm{II}}|\psi_{\mathrm{II}}\rangle \tag{1.2}$$

its time evolution is found from Schrödinger's equation

$$\mathrm{i}\hbar\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = \hat{H}|\psi\rangle \tag{1.3}$$

as

$$\begin{aligned}|\psi_{\mathrm{II}}(t)\rangle &= \exp\left[\frac{-\mathrm{i}\hat{H}t}{\hbar}\right]|\psi_{\mathrm{II}}(0)\rangle \\ &= \exp\left[\frac{-\mathrm{i}E_{\mathrm{II}}t}{\hbar}\right]|\psi_{\mathrm{II}}(0)\rangle\end{aligned} \tag{1.4}$$

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and a similar result holds for the lower state $|\psi_{\rm I}\rangle$. Therefore, if the electron undergoes some transition from the upper to the lower level, its intermediate quantum state should be some linear combination of the kind

$$
\begin{aligned}
|\psi(t)\rangle &= C_{\rm I}(t)|\psi_{\rm I}(t)\rangle + C_{\rm II}(t)|\psi_{\rm II}(t)\rangle \\
&= C_{\rm I}{\rm e}^{-{\rm i}E_{\rm I}t/\hbar}\,|\psi_{\rm I}\rangle + C_{\rm II}{\rm e}^{-{\rm i}E_{\rm II}t/\hbar}\,|\psi_{\rm II}\rangle
\end{aligned}
\tag{1.5}
$$

so that the density $\rho(\boldsymbol{x},t) = \overset{*}{\psi}\cdot\psi(\boldsymbol{x},t)$ acquires an oscillating interference part $\rho_{\rm os}$, i.e.

$$\rho(\boldsymbol{x},t) = \rho_0(\boldsymbol{x}) + \rho_{\rm os}(\boldsymbol{x},t) \tag{1.6a}$$

$$\rho_0(\boldsymbol{x}) = |C_{\rm I}|^2\,|\psi_{\rm I}(\boldsymbol{x})|^2 + |C_{\rm II}|^2\,|\psi_{\rm II}(\boldsymbol{x})|^2 \tag{1.6b}$$

$$\rho_{\rm os}(\boldsymbol{x},t) = \overset{*}{C}_{\rm I}C_{\rm II}{\rm e}^{-{\rm i}\omega t}\overset{*}{\psi}_{\rm I}(\boldsymbol{x})\,\psi_{\rm II}(\boldsymbol{x}) + \text{c.c.}\,. \tag{1.6c}$$

In view of this result, it now appears as a matter of course that the oscillatory part $\rho_{\rm os}$ will act as a source for radiation with just the frequency ω given by Bohr's formula (1.1).

This paper is concerned with the relativistic generalization of the splitting (1.6), i.e. the separation of the wavefunction ψ into a rapidly varying part, responsible for radiation, and into a slowly varying part which describes the average (i.e. smoothed) motion of the radiating particle *in the relativistic domain.* Here we expect a much higher frequency ω, because a particle is in general mixed up with its anti-particle and the characteristic frequency for particle annihilation is in the order of magnitude $\omega \approx 2Mc^2/\hbar$ (where M is the physical mass of the particle). In order to make our intention more concrete, we give a short demonstration of this effect by considering a Dirac spinor field $\psi(x)$ over Minkowski space-time. We require it to satisfy the relativistic generalization of Schrödinger's equation (1.3)

$$\mathrm{i}\hbar c\partial_\mu\psi = \mathcal{H}_\mu\psi \tag{1.7}$$

which has recently been shown [2] to be equivalent to Dirac's equation

$$\mathrm{i}\hbar\gamma^\mu\partial_\mu\psi = Mc\psi \tag{1.8}$$

if one imposes the following condition (among others) upon the Hamiltonian 1-form $\mathcal{H}_\mu$

$$\gamma^\mu\mathcal{H}_\mu = Mc^2\,. \tag{1.9}$$

In short, the Dirac wave equation $\psi(x)$ (more properly: $|\psi(x)\rangle$) is considered here as the section of a *four-dimensional* vector bundle with typical fibre $\mathbf{C}^4$ over space-time as the corresponding base manifold; quite similarly as $|\psi(t)\rangle$ in Schrödinger's non-relativistic theory (1.3) may be considered as some section of an *infinite-dimensional* vector bundle with the Newtonian time axis $\mathbf{R}^1$ as the corresponding base space. Therefore, whereas the Schrödinger Hamiltonian $\hat{H}$ (1.3) is in general a (time-dependent) operator over an (in)finite-dimensional Hilbert space, our Hamiltonian $\mathcal{H}_\mu$ (1.7) is a $\mathcal{G}\ell(4,\mathbf{C})$-valued 1-form and is in general space-time dependent rather than time dependent. This formal transition from the non-relativistic Schrödinger theory to our present relativistic approach is best exemplified by a treatment of the *free particle.*

For a free particle with four-momentum p_μ, wave vector k_μ and four velocity b_μ ($\leadsto b^\mu b_\mu = 1$), i.e.

$$p_\mu = \hbar k_\mu = Mcb_\mu \tag{1.10}$$

the relativistic Hamiltonian $\mathcal{H}_\mu$ is readily found as

$$\mathcal{H}_\mu = Mc^2 b_\mu (b_\lambda \gamma^\lambda) \tag{1.11}$$

and consequently the relativistic wavefunction ψ reads in a certain analogy to the non-relativistic case (1.4)

$$\psi(x) = \exp\left[-\mathrm{i}\frac{\mathcal{H}_\mu x^\mu}{\hbar c}\right] \cdot \psi(0)\,. \tag{1.12}$$

Here, the initial value $\psi(0)$ at the origin of Minkowskian space-time (without loss of generality) is an arbitrary element of the typical fibre $\mathbf{C}^4$. However, the difference is that the non-relativistic time-evolution operator

$$\hat{U}(t) = \exp\left[-\frac{\mathrm{i}\hat{H}t}{\hbar}\right] \tag{1.13}$$

in Schrödinger's theory has now been generalized into a relativistic space-time evolution operator

$$\mathcal{U}(x) = \exp\left[-\mathrm{i}\frac{\mathcal{H}_\mu x^\mu}{\hbar c}\right] \tag{1.14}$$

which propagates the initial value of the wavefunction $\psi(0)$ from the origin of Minkowskian space-time into the whole 4-manifold. Observe, however, that the 'unitarity' ($\bar{\mathcal{U}} \equiv \mathcal{U}^{-1}$) of the relativistic evolution operator $\mathcal{U}$ (1.14) is incidental here, because the relativistic Hamiltonian $\mathcal{H}_\mu$ will in general not be 'Hermitian' ($\bar{\mathcal{H}}_\mu \neq \mathcal{H}_\mu$), in contrast to Schrödinger's operator $\hat{H} \equiv \hat{H}^+$. (As a demonstration, see the treatment of the relativistic hydrogen atom [2]).

After this side-step to look at the formal analogies between the non-relativistic and the relativistic approaches, we return to our original problem of identifying slow and rapid variables in the relativistic domain.

Looking at the current density $j_\mu = \bar{\psi} \cdot \gamma_\mu \cdot \psi$ generated by the solution $\psi(x)$ (1.12), we readily find [2]

$$\begin{aligned} j_\mu \to {}^{(0)}j^\mu &= \bar{\psi}(0) \cdot \bar{\mathcal{U}}(x) \cdot \gamma^\mu \cdot \mathcal{U}(x) \cdot \psi(0) \\ &= \big(b_\lambda j^\lambda(0)\big)b^\mu + \cos\chi\big[j^\mu(0) - \big(b_\lambda j^\lambda(0)\big)b^\mu\big] + 4\sin\chi\, b_\lambda S^{\lambda\mu}(0)\,. \end{aligned} \tag{1.15}$$

Here, we have anticipated the polarization tensor $S_{\mu\nu}$ which is usually defined as

$$S_{\mu\nu} := \tfrac{1}{2}\mathrm{i}\bar{\psi} \cdot \Sigma_{\mu\nu} \cdot \psi \qquad \left(\Sigma_{\mu\nu} = \tfrac{1}{4}[\gamma_\mu, \gamma_\nu]\right) \tag{1.16}$$

with its initial value $S_{\mu\nu}(0)$ being built up by the initial value $\psi(0)$ of the wavefunction in an obvious way. The interesting point here is the phase χ emerging in the current (1.15)

$$\begin{aligned} \chi(x) &= 2mb_\nu x^\nu \\ \rightsquigarrow \dot{\chi} &\equiv b^\nu \partial_\nu \chi = 2m\left(\equiv 2\frac{Mc}{\hbar}\right). \end{aligned} \tag{1.17}$$

Obviously, the quantity $\dot{\chi}$ is the relativistic generalization of the frequency ω (1.6c) and is just in the expected order of magnitude (the Compton-frequency $\omega_c = 2Mc^2/\hbar \approx 10^{21}\,\mathrm{s}^{-1}$). Thus, we see that—in the relativistic case—it is already the *free* particle which in general develops rapid oscillations around its average translatory motion characterized by the constant four-velocity b_μ ($\rightsquigarrow$ first term in (1.15)). This result corresponds to what is known in the literature as *Schrödinger's Zitterbewegung* [3–5]. Similarly to the non-relativistic case, this trembling motion is absent whenever the particle is in an eigen-state $\psi_p(x)$ of the Hamiltonian, i.e. (cf (1.12))

$$\begin{aligned} \mathcal{H}_\mu \psi(0) &= \pm Mc^2 b_\mu \psi(0) \\ \rightsquigarrow \psi_p(x) &= \exp\left(\mp \mathrm{i}\frac{p_\mu x^\mu}{\hbar}\right)\psi(0)\,. \end{aligned} \tag{1.18}$$

In this case the current simply becomes

$$^{(0)}j_\mu = (\bar{\psi}_p \cdot \psi_p) b_\mu \equiv \rho_p b_\mu \tag{1.19}$$

as is normally expected for a particle in uniform motion.

After these preparations, we are now able to define more precisely the problem to be dealt with subsequently: also expecting that the other physical densities of Dirac's spinor field (such as pseudo density $\tilde{\rho}$, axial current $\tilde{\gamma}_\mu$ and polarization $S_{\mu\nu}$) will split up into violently oscillating and slowly varying parts, we want to identify these kinematically different parts and discuss their properties. Here we want to abandon the restrictions to a free particle and consider the quite general case. Especially, one wants to know whether the well known *Gordon decomposition* [5, 6] really succeeds in separating the slow and rapid parts of the physical densities ($\rightsquigarrow$ it does not, see below). Such an undertaking would be facilitated greatly if one could identify the slow and rapid variables already within the wavefunction ψ itself, which we are now looking for.

2. Spinor kinematics

We consider the wavefunction $\Psi(x)$ as some (local or global) section of a four-dimensional, complex vector bundle Ψ_4. Thus, $\Psi(x)$ mediates a (local) map from any point x of our pseudo-Riemannian space time $\mathbf{V}_4$ as base space (with metric $G_{\mu\nu}$) into the typical fibre $\mathbf{C}^4$ of the bundle Ψ_4, i.e.

$$\Psi:\ x \to \Psi(x) \in \mathbf{C}^4 \qquad x \in \mathbf{V}_4\,.$$

If the physical situation admits only local sections $\Psi(x)$, we have to cover our space-time $\mathbf{V}_4$ with several patches and apply a gauge transformation on the wavefunction Ψ in the intersections

$$\Psi' = \mathcal{S}^{-1} \cdot \Psi\,. \tag{2.1}$$

Here, the group element $\mathcal{S}$ is a member of the structure group $Spin(1,3)$ for the vector bundle Ψ_4 [7]. Any such section $\Psi(x)$ is referred to as an associated section $\mathrm{B}(x)$ of the principal bundle $\mathcal{B}_4$ of orthonormal pseudo-frames over the same base space $\mathbf{V}_4$.

This implies that any gauge transformation (2.1) of the spinor Ψ is accompanied by a homomorphic transformation of the tetrad constituents $\mathcal{B}_{\alpha\mu}(x)$ of $\mathrm{B}(x)$

$$\begin{aligned} &\mathcal{B}'_{\alpha\mu} = \mathcal{B}_{\beta\mu}\Lambda^{\beta}{}_{\alpha} \\ &\left(\mathcal{B}_{\alpha\mu}\mathcal{B}^{\alpha}{}_{\nu} = G_{\mu\nu}\,. \qquad \mathcal{B}_{\alpha\mu}\mathcal{B}_{\beta}{}^{\mu} = g_{\alpha\beta}\right) \end{aligned} \tag{2.2}$$

where Λ is an element of the Lorentz group $SO(1,3)$, the invariance group for the Minkowskian metric $g_{\alpha\beta}$. The homomorphic relationship of both gauge elements $\mathcal{S}$ and Λ is frequently expressed through

$$\mathcal{S}^{-1}\cdot\hat{\gamma}^{\alpha}\cdot\mathcal{S} = \Lambda^{\alpha}{}_{\beta}\,\hat{\gamma}^{\beta} \tag{2.3}$$

where $\hat{\gamma}^{\alpha}$ are the ordinary Dirac matrices acting as operators on the typical fibre $\mathbf{C}^4$ of Ψ_4. As a consequence, the *physical densities* carried by the Dirac spinor field are left gauge invariant

$$\rho'(x) = \rho(x)\left(:= \bar{\Psi}\cdot\Psi(x)\right) \tag{2.4a}$$

$$\tilde{\rho}'(x) = \tilde{\rho}(x)\left(:= \bar{\Psi}\cdot\varepsilon\cdot\Psi(x)\right) \tag{2.4b}$$

$$\left(\varepsilon := \frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}\hat{\gamma}^{\alpha}\hat{\gamma}^{\beta}\hat{\gamma}^{\gamma}\hat{\gamma}^{\delta}\right) \tag{2.4c}$$

$$j'_{\mu}(x) = j_{\mu}(x)\left(:= \mathcal{B}_{\alpha\mu}\bar{\Psi}\cdot\hat{\gamma}^{\alpha}\cdot\Psi \equiv \bar{\Psi}\cdot\gamma_{\mu}\cdot\Psi\right) \tag{2.4d}$$

$$\tilde{j}'_{\mu}(x) = \tilde{j}_{\mu}(x)\left(:= i\bar{\Psi}\cdot\varepsilon\gamma_{\mu}\cdot\Psi \equiv i\bar{\Psi}\cdot\tilde{\gamma}_{\mu}\cdot\Psi\right). \tag{2.4e}$$

(Clearly, one is willing to consider only gauge invariant objects as physical densities!)

Up to now we have done nothing else than collecting the standard knowledge about a spinor field Ψ, but now we use a new argument in order to reparametrize Ψ in physical terms. We start from the observation [8, 9] that the principal $SO(1,3)$ bundle B_4 admits an $SO(3)$ reduced subbundle $\bar{\mathcal{B}}_4$, which is equivalent to the existence of a *global* unit section $b_{\mu}(x)$ $(b_{\mu}b^{\mu} = +1)$ of the tangent bundle T_4 of space-time V_4. Now we *identify* this distinguished vector b_{μ} with the average four-velocity field of the Dirac particle (cf the heuristic arguments for the free particle). Once the full gauge symmetry (2.1), (2.2) has been broken down in this way to the ordinary rotation group $SO(3) \subset SO(1,3)$ (respectively $SU(2) \subset Spin(1,3)$), we further identify the time-like tetrad vector $\mathcal{B}_{0\mu}$ with that particle velocity field b_{μ}, i.e. $b_{\mu} \equiv \mathcal{B}_{0\mu}$. Next, we try the following general ansatz for the spinor Ψ relative to that distinguished tetrad $\mathcal{B}_{\alpha\mu} = \{b_{\mu}, \mathcal{B}_{j\mu};\ (j = 1,2,3)\}$

$$\Psi = \sqrt{\rho}\left[\cosh\kappa(a_{\uparrow}u_{+} + a_{\downarrow}u_{-}) + \mathrm{e}^{\mathrm{i}\chi}\sinh\kappa(b_{\downarrow}v_{-} + b_{\uparrow}v_{+})\right]. \tag{2.5}$$

In what follows, we will readily explain and discuss this ansatz in detail.

The expression (2.5) shows the (local) decomposition of Ψ with respect to some orthonormal spinor basis $\{u_{\pm}, v_{\mp}\}$ for the typical fibre $\mathbf{C}^4$, i.e.

$$\bar{u}_{+}\cdot u_{+} = \bar{u}_{-}\cdot u_{-} = -\bar{v}_{-}\cdot v_{-} = -\bar{v}_{+}\cdot v_{+} = +1 \tag{2.6a}$$

$$\bar{u}_{+}\cdot u_{-} = \bar{u}_{+}\cdot v_{-} = \cdots = 0\,. \tag{2.6b}$$

Its coefficients $\{a_\uparrow, a_\downarrow\}$ and $\{b_\downarrow, b_\uparrow\}$ form two 2-spinors α, β

$$\alpha = \begin{pmatrix} a_\uparrow \\ a_\downarrow \end{pmatrix} \qquad \beta = \begin{pmatrix} b_\downarrow \\ b_\uparrow \end{pmatrix} \tag{2.7}$$

which transform according to the fundamental representation $\left({}^{(2)}\mathcal{S}\right)$ of the remaining gauge degree of freedom $SU(2)$, i.e.

$$\alpha' = \left({}^{(2)}\mathcal{S}\right)^{-1} \cdot \alpha \tag{2.8a}$$

$$\beta' = \left({}^{(2)}\mathcal{S}\right)^{-1} \cdot \beta \tag{2.8b}$$

$$\left({}^{(2)}\mathcal{S} \in SU(2) \subset Spin(1,3)\right).$$

Both 2-spinors α, β are required to be normalized to unity, i.e.

$$\begin{aligned} 1 &= \alpha^+ \cdot \alpha \ \left(\equiv \overset{*}{a}_\uparrow\, a_\uparrow + \overset{*}{a}_\downarrow\, a_\downarrow \right) \\ 1 &= \beta^+ \cdot \beta \end{aligned} \tag{2.9}$$

and consequently they represent together $2 \cdot (4-1) - 1 = 5$ field degrees of freedom because we have equipped their overall relative phase χ with an extra degree of freedom. Clearly, this angular variable χ will be regarded as the relativistic analogue of the non-relativistic phase difference of the two stationary states (cf (1.5), (1.6)) and therefore will be regarded as a 'rapid' variable, in contrast to the 'slow' spinors α, β. The remaining two variables for our re-parametrization of the 8-parametric wave function Ψ are the 'intrinsic velocity' κ and the density $\rho (= \bar{\Psi} \cdot \Psi)$; the latter is assumed to be positive everywhere (otherwise one interchanges the hyperbolic functions). Both variables κ and ρ are considered as slow so that the phase χ is left as the only rapid variable. This means that the intrinsic velocity κ, measuring the relative independence of the actual current j_μ from the average velocity b_μ (see below), is changing only slowly and consequently the trembling effect due to the *Schrödinger Zitterbewegung* is expected to arise from rapid changes of the spatial direction of the current j_μ. Clearly this process must be governed by the rapid phase χ.

For the verification of a such a concrete picture, one has to explicitly compute the physical densities (2.4) by means of the wave function Ψ (2.5). This will be done in two steps, namely by first computing these densities in the distinguished reference frame ($\mathcal{B}_{\alpha\mu} \Rightarrow \{b_\mu, \mathcal{B}_{j\mu}\}$) and then pulling the results back to a coordinate basis, as indicated in equation (2.4*d*).

2.1. Calculation in the distinguished reference frame

During the first step, there emerge some $SO(3)$ invariants, such as

$$z_+ = \tfrac{1}{2}(\alpha^+ \cdot \beta + \beta^+ \cdot \alpha) \tag{2.10a}$$

$$z_- = \tfrac{1}{2}(\alpha^+ \cdot \beta - \beta^+ \cdot \alpha) \tag{2.10b}$$

as well as some $SO(3)$ vectors involving the Pauli spin matrices σ^k $(k = 1, 2, 3)$

$$\hat{m}^k = \tfrac{1}{2}\left(\alpha^+ \cdot \sigma^k \cdot \beta + \beta^+ \cdot \sigma^k \cdot \alpha\right) \tag{2.11a}$$

$$\hat{n}^k = \tfrac{1}{2\mathrm{i}}\left(\alpha^+ \cdot \sigma^k \cdot \beta - \beta^+ \cdot \sigma^k \cdot \alpha\right) \tag{2.11b}$$

$$\hat{\vec{q}}^k = \tfrac{1}{2}\left(\alpha^+ \cdot \sigma^k \cdot \alpha + \beta^+ \cdot \sigma^k \cdot \beta\right) \tag{2.11c}$$

$$\hat{\vec{l}}^k = \tfrac{1}{2}\left(\alpha^+ \cdot \sigma^k \cdot \alpha - \beta^+ \cdot \sigma^k \cdot \beta\right). \tag{2.11d}$$

One is easily convinced by means of the two-dimensional analogue of the relationship (2.3)

$$ {}^{(2)}\mathcal{S}\cdot\sigma^k\cdot\left({}^{(2)}\mathcal{S}^{-1}\right)=(S^{-1})^k{}_j\sigma^j \qquad {}^{(2)}\mathcal{S}\in SU(2) \qquad S\in(3) \tag{2.12} $$

that these $SO(3)$ vectors correctly transform under a gauge rotation of the reference triad

$$ \mathcal{B}'_{j\mu}=\mathcal{B}_{k\mu}S^k{}_j \tag{2.13} $$

i.e.

$$ \hat{m}^{k'}=(S^{-1})^k{}_j\,\hat{m}^j \qquad \text{etc.} \tag{2.14} $$

Moreover, their products are determined by the scalars (2.10) in the following way

$$ \hat{m}^k\hat{m}_k=z_-^2-1 \tag{2.15a} $$

$$ \hat{n}^k\hat{n}_k=z_+^2-1 \tag{2.15b} $$

$$ \hat{m}^k\hat{n}_k=-z_+z_- \tag{2.15c} $$

$$ \hat{\hat{q}}^k\hat{\hat{q}}_k=-(z_+^2+z_-^2) \tag{2.15d} $$

$$ \hat{\hat{l}}^k\hat{\hat{l}}_k=-1+z_+^2+z_-^2 \tag{2.15e} $$

$$ \hat{n}^k\hat{\hat{q}}_k=-z_- \tag{2.15f} $$

$$ \hat{m}^k\hat{\hat{q}}_k=-z_+ \tag{2.15g} $$

$$ \hat{\hat{l}}^k\hat{n}_k=\hat{\hat{l}}^k\hat{m}_k=\hat{\hat{l}}^k\hat{\hat{q}}_k=0\,. \tag{2.15h} $$

Here, the last relation (2.15h) shows that all three vectors $\hat{n}^k,\hat{m}^k,\hat{\hat{q}}^k$ are orthogonal to the fourth vector $\hat{\hat{l}}^k$; therefore we can try to write $\hat{\hat{q}}^k$ as a combination of the first two vectors $\hat{n}^k,\hat{m}^k$ and find

$$ \hat{\hat{q}}^k=z_+\hat{m}^k+z_-\hat{n}^k\,. \tag{2.16} $$

Similarly the fourth vector $\hat{\hat{l}}^k$ reads in terms of $\hat{n}^k,\hat{m}^k$

$$ \hat{\hat{l}}^j=\varepsilon^j{}_{kl}\,\hat{m}^k\hat{n}^l\,. \tag{2.17} $$

Further, since $\hat{m}^k$ and $\hat{n}^k$ are not orthogonal (cf (2.15c)), it may be convenient to choose an orthogonal basis in their two-plane by looking for the orthogonal complement $\hat{\hat{p}}^k$ of the vector $\hat{\hat{q}}^k$ (2.16), i.e. we put

$$ \hat{\hat{p}}^k=z_-\hat{m}^k-z_+\hat{n}^k \tag{2.18a} $$

such that

$$ \hat{\hat{p}}^k\hat{\hat{q}}_k=0 \tag{2.18b} $$

and the length of the new vector $\hat{\tilde{p}}^k$ is then found as

$$\hat{\tilde{p}}^k \hat{\tilde{p}}_k = (z_-^2 + z_+^2)(-1 + z_-^2 + z_+^2) . \tag{2.19}$$

Comparing the last equation to (2.15*d*), it seems sensible to renormalize both vectors $\hat{\tilde{q}}_k$ and $\hat{\tilde{p}}_k$ as

$$\hat{\tilde{g}}^k = z^{-1} \hat{\tilde{q}}^k \tag{2.20a}$$

$$\hat{\tilde{f}}^k = z^{-1} \hat{\tilde{p}}^k \tag{2.20b}$$

with

$$z := \sqrt{z_+^2 + z_-^2} \tag{2.21}$$

so that $\hat{\tilde{g}}^k$ becomes a unit vector

$$\hat{\tilde{g}}^k \hat{\tilde{g}}_k = -1 \tag{2.22}$$

and $\hat{\tilde{f}}^k$ acquires the same length as $\hat{\tilde{l}}^k$ (2.15*e*)

$$\hat{\tilde{f}}^k \hat{\tilde{f}}_k = \hat{\tilde{l}}^k \hat{\tilde{l}}_k = -1 + z^2 . \tag{2.23}$$

Consequently, equation (2.17) for the reconstruction of $\hat{\tilde{l}}^k$ from the new orthogonal basis reads now

$$\hat{\tilde{l}}^k = \varepsilon^k{}_{jl} \hat{\tilde{f}}^j \hat{\tilde{g}}^l . \tag{2.24}$$

We have now collected all relevant $SO(3)$ gauge objects, which we need for the construction of the $SO(1,3)$ invariant physical densities (2.4) and can now proceed to the second step.

2.2. *Transfer to a coordinate basis*

These densities can readily be written down after referring the $SO(3)$ objects back to some coordinate basis, e.g.

$$\tilde{g}_\mu = \mathcal{B}_{k\mu} \hat{\tilde{g}}^k \tag{2.25}$$

or (cf (2.24))

$$\tilde{l}_\nu = b^\mu \varepsilon_{\mu\nu\lambda\sigma} \tilde{f}^\lambda \tilde{g}^\sigma \qquad \text{etc.} \tag{2.26}$$

We thus find for the desired densities

$$\tilde{\rho} = \rho z \cdot \sinh 2\kappa \cdot \sin(\chi + \zeta) \tag{2.27a}$$

$$j_\mu = \rho \big\{ \cosh 2\kappa \cdot b_\mu + \sinh 2\kappa [\tilde{g}_\mu \cos(\chi + \zeta) + \tilde{f}_\mu \sin(\chi + \zeta)] \big\} \tag{2.27b}$$

$$\tilde{j}_\mu = -\rho \big\{ z \sinh 2\kappa \cdot \cos(\chi + \zeta) \cdot b_\mu + z \cosh 2\kappa \cdot \tilde{g}_\mu + \tilde{l}_\mu \big\} \tag{2.27c}$$

$$S_{\mu\nu} = -\tfrac{1}{2} \rho \sinh 2\kappa \big[b_{[\mu} \tilde{g}_{\nu]} \sin(\chi + \zeta) - b_{[\mu} \tilde{f}_{\nu]} \cdot \cos(\chi + \zeta) \big] + \tfrac{1}{4} \rho\, b^\lambda \varepsilon_{\lambda\sigma\mu\nu} \big(z \tilde{g}^\sigma + \cosh 2\kappa \cdot \tilde{l}^\sigma \big) . \tag{2.27d}$$

Here, we have introduced the phase shift ζ as a *slow* variable, in contrast to the *rapid* phase χ, because ζ is constructed by means of the slow two-spinors α, β through (cf (2.10))

$$\cos\zeta = \frac{z_+}{z} \qquad \sin\zeta = \frac{z_-}{z}. \tag{2.28}$$

Our result (2.27) is now considered as the desired relativistic generalization of our starting point (1.6). There is a nice consistency check for this result, because the densities $\rho, \tilde{\rho}, j_\mu, \tilde{j}_\mu, S_{\mu\nu}$ must obey the recently discovered identities [10]

$$j^\mu j_\mu \equiv -\tilde{j}^\mu \tilde{j}_\mu \equiv \rho^2 + \tilde{\rho}^2 \tag{2.29a}$$

$$j^\mu \tilde{j}_\mu \equiv 0 \tag{2.29b}$$

$$S_{\mu\nu} j^\mu \equiv \tfrac{1}{4}\tilde{\rho}\, \tilde{j}_\nu \tag{2.29c}$$

$$\overset{*}{S}_{\mu\nu}\, j^\mu \equiv \tfrac{1}{4}\rho\, \tilde{j}_\nu \tag{2.29d}$$

$$S_{\mu\nu}\tilde{j}^\mu \equiv \tfrac{1}{4}\tilde{\rho}\, j_\nu\,. \tag{2.29e}$$

These identities are easily verified by means of the scalar products (2.15), which read in a coordinate basis (cf e.g. (2.23))

$$G^{\mu\nu}\tilde{f}_\mu \tilde{f}_\nu \equiv \tilde{f}^\mu \tilde{f}_\mu = \tilde{l}^\mu \tilde{l}_\mu = -1 + z^2 \qquad \text{etc.} \tag{2.30}$$

(Observe that the 4 vectors $\{b_\mu, \tilde{l}_\mu, \tilde{g}_\mu, \tilde{f}_\mu\}$ form an *orthogonal* tetrad.)

3. Applications

After the physical densities of Dirac's spinor field have been kinematically split up into their rapidly and slowly varying parts, one wants to know what the consequences of this splitting are for various physical effects. In this paper, we restrict ourselves to three items, namely the questions:

(i) Whether that kinematical splitting is identical to the well-known Gordon decomposition of the current j_μ.
(ii) Whether both the magnetic and electric dipole densities are affected by these rapid oscillations in a similar way.
(iii) Why a *closed* FRW universe is forbidden by the Dirac–Einstein equations.

However before going into the details let us confirm that the splitting achieved agrees with the previous free-particle results (1.15)–(1.17), which have been obtained by solving Dirac's equation directly.

3.1. Free particle in flat space-time

For this special situation, the principal bundle $\mathcal{B}_4$ admits a *global* section $\mathrm{B}(x)$ and the corresponding solution $\Psi(x)$ (1.12) leads to the current j_μ (1.15). It is only in this special case that, by virtue of the global teleparallelism, the initial values $j^\mu(0)$ and $S^{\mu\nu}(0)$ at the origin could be propagated into the whole Minkowski space. On the other hand, the *general* shape of j_μ and $S_{\mu\nu}$ is exhibited in equations (2.27) and consequently both results must fit together. Indeed, taking the general form of the initial values $j^\mu(0)$ and $S^{\mu\nu}(0)$ from (2.27)

$$\begin{aligned} b_\lambda j^\lambda(0) &= \rho(0)\cosh 2\kappa(0) \\ b_\lambda S^{\lambda\mu}(0) &= -\tfrac{1}{4}\rho(0)\sinh 2\kappa(0)\cdot\left(\tilde{g}^\mu(0)\sin[\chi(0)+\zeta(0)] - \tilde{f}^\mu(0)\cos[\chi(0)+\zeta(0)]\right)\end{aligned} \tag{3.1}$$

and substituting this into (1.15) just reproduces that free particle result (1.15) (respectively (2.27*b*)), namely

$$j^\mu(x) = \rho(0)\left[\cosh 2\kappa(0) b^\mu + \sinh 2\kappa(0)\cdot(\tilde{g}^\mu(0)\cdot\cos\chi(x) + \tilde{f}^\mu(0)\sin\chi(x))\right] \tag{3.2}$$

but with the obvious replacement

$$\chi(0)+\zeta(0) \Rightarrow \chi(x) = \chi(0)+\zeta(0)+2mb_\nu x^\nu\,. \tag{3.3}$$

Thus we see that the (free-particle) Hamiltonian $\mathcal{H}_\mu$ (1.11) does nothing else other than drive the rapid phase χ according to (3.3), leaving constant the slow variables $\rho, \kappa, \zeta, \tilde{g}_\mu, \tilde{f}_\mu$. Of course the latter variables are expected to vary too, when the free particle enters some force field, but not so rapidly as does the phase χ. Tacitly it is assumed here that a clear distinction can always be made between 'slow' and 'rapid' variations of Ψ, at least when the external forces are weak and slow enough!

3.2. Gordon decomposition

The free-particle result (3.2) is also a neat demonstration of the trembling effect: the current j_μ consists of an average translational motion along the b_μ direction and of a rapid trembling in the orthogonal 2-plane ($b^\mu\tilde{f}_\mu = b^\mu\tilde{g}_\mu = 0$) with the Compton frequency $\omega_c = \dot{\chi}$ (1.17). On the other hand, the well-known Gordon decompositon of the current j_μ

$$j_\mu = {}^{(c)}j_\mu + {}^{(p)}j_\mu \tag{3.4}$$

is intended to split up the current into a 'convection' part (c) and a 'polarization' part (p), so that one is tempted to identify this convection part with that average component of the motion ($\sim b_\mu$) and similarly the polarization part may be intuitively associated with the remaining trembling component. Indeed, at first glance this identification is perfectly supported by the free particle case: splitting up the velocity operator [2] $\gamma_\mu = \beta_\mu + \pi_\mu$ into its convective part β_μ

$$\beta^\mu = \frac{1}{2Mc^2}\left(\bar{\mathcal{H}}^\mu + \mathcal{H}^\mu\right) \tag{3.5}$$

and its polarization part π_μ

$$\pi^\mu = \frac{1}{Mc^2}\left(\bar{\mathcal{H}}_\lambda\cdot\Sigma^{\lambda\mu} + \Sigma^{\mu\lambda}\cdot\mathcal{H}_\lambda\right) \tag{3.6}$$

we readily find by means of the free-particle Hamiltonian $\mathcal{H}_\mu$ (1.11)

$$^{(c)}j_\mu = \bar{\Psi} \cdot \beta_\mu \cdot \Psi = \rho(0) \cosh 2\kappa(0) \cdot b_\mu \tag{3.7a}$$

$$^{(p)}j^\mu = \bar{\Psi} \cdot \pi^\mu \cdot \Psi = \rho(0) \sinh 2\kappa(0)\big(\tilde{g}^\mu(0) \cos\chi(x) + \tilde{f}^\mu(0) \sin\chi(x)\big). \tag{3.7b}$$

Thus, this free-particle result appears to meet very well with the intuitive expectation mentioned above.

However, this quick success does not withstand a second scrutiny. The reason is that the free-particle case is too specific. Therefore let us consider a more complicated situation by putting the Dirac particle into a FRW universe [10]. Here, we have to use the cosmic form for the Hamiltonian $\mathcal{H}_\mu$, which additionally depends upon the radius $\mathcal{R}$ of the universe, the Hubble expansion rate H $(= \dot{\mathcal{R}}/\mathcal{R})$ and upon the foliation index σ $(= 0, \pm 1)$. For this situation the splitting analogous to (3.7) for the current looks as follows

$$\begin{aligned} ^{(c)}j_\mu = b_\mu \rho \cosh 2\kappa \left(1 + \frac{3\sigma}{2m\mathcal{R}} \cos\chi\right) - \frac{\sigma}{2m\mathcal{R}} \rho \sinh 2\kappa\big(\cos\zeta \cdot \tilde{g}_\mu + \sin\zeta \cdot \tilde{f}_\mu\big) \\ + \frac{H}{2m} \rho \sinh 2\kappa\big(\tilde{g}_\mu \cdot \sin(\chi + \zeta) - \tilde{f}_\mu \cos(\chi + \zeta)\big) \end{aligned} \tag{3.8}$$

and similarly for the polarization current

$$\begin{aligned} ^{(p)}j_\mu = -\frac{3\sigma}{2m\mathcal{R}} \rho \cosh 2\kappa \cdot \cos\chi\, b_\mu + \rho \sinh 2\kappa\big(\tilde{g}_\mu \cos(\chi + \zeta) + \tilde{f}_\mu \sin(\chi + \zeta)\big) \\ + \frac{\sigma}{2m\mathcal{R}} \rho \sinh 2\kappa\big(\tilde{g}_\mu \cos\zeta + \tilde{f}_\mu \sin\zeta\big) \\ - \frac{H}{2m} \rho \sinh 2\kappa\big(\tilde{g}_\mu \sin(\chi + \zeta) - \tilde{f}_\mu \cos(\chi + \zeta)\big). \end{aligned} \tag{3.9}$$

Thus we see that the naive supposition is *correct only for a flat space-time*, i.e. for a flat universe $(\sigma = 0)$ with vanishing Hubble expansion rate $H \equiv 0$ ($\rightsquigarrow$ constant 'radius' $\mathcal{R}$ of the universe). In a closed $(\sigma = -1)$ or open $(\sigma = +1)$ universe, the result (3.8) says that the 'smooth' component ($\sim b_\mu$) of $^{(c)}j_\mu$ acquires a rapidly oscillating correction term ($\sim \cos\chi$) and a slow transverse component (being free of the rapid angle χ). But evidently these two terms become relevant only in a *small* universe $(m\mathcal{R} \equiv Mc\mathcal{R}/\hbar \sim 1)$, i.e. when the size of the universe is in the order of magnitude of the Compton wave length (m^{-1}) of the Dirac particle! If the universe is large enough $(m\mathcal{R} \gg 1)$, these two terms become negligible and the convection current $^{(c)}j_\mu$ (3.8) really approaches its flat space-time counterpart (3.7*a*), provided the expansion rate $H \equiv \dot{\mathcal{R}}/\mathcal{R}$ is small enough compared to the inverse Compton wavelength m $(H \ll m)$. Similar arguments hold for the comparison of the polarization current $^{(p)}j_\mu$ (3.9) in curved space-time, to its flat space-time analogue (3.7*b*).

Thus the general conclusion from these results is that the Gordon decomposition separates the slow and rapid motions only when the external forces are small enough to consider the particle as travelling freely in an approximately flat space-time! More precisely: as soon as the external forces or the background geometry produce essential changes of motion to the Compton length and time scale, the Gordon decomposition becomes kinematically meaningless. (Of course, these are also the limitations for any single-particle theory.)

3.3. Dipole densities

The kinematic splitting of the physical densities is also relevant for the polarization tensor $S_{\mu\nu}$. As is well known, this tensor accounts for the magnetic and electric dipole properties of a Dirac particle; however it depends strongly upon the observer as to what he sees as magnetic and as electric. Thus, in the present context, we may ask whether there is some observer who sees one of the two dipole densities being completely free of the rapid variations inherent in the wave function Ψ.

Let the four-velocity of the observer be V_μ ($V^\mu V_\mu = +1$), then the physical polarization $M_{\mu\nu}$ is split up into its magnetic and electric parts through

$$M_{\mu\nu} \equiv \frac{2e}{m} S_{\mu\nu} = {}^{(\mathrm{el})}M_\mu V_\nu - {}^{(\mathrm{el})}M_\nu V_\mu + \varepsilon_{\mu\nu\lambda\sigma} V^{\lambda\,(\mathrm{m})}M^\sigma \tag{3.10}$$

where ${}^{(\cdot)}M_\mu$ denotes the electric (el) and magnetic (m) dipole densities, respectively. As a matter of principle, there are two manifest choices for the observer in question:

(i) The first observer is co-moving with the current j_μ, i.e. his four-velocity $V_\mu \Rightarrow u_\mu$ ($u^\mu u_\mu = +1$) is proportional to the current j_μ ($= \sqrt{\rho^2 + \tilde{\rho}^2}\, u_\mu$). For this choice, the dipole densities have been found to be proportional to the polarization vector s_μ ($u^\mu s_\mu = 0$) [10]

$$ {}^{(\mathrm{el})}M_\mu = -\frac{e\hbar}{2Mc} \tilde{\rho}\, s_\mu \tag{3.11a}$$

$$ {}^{(\mathrm{m})}M_\mu = -\frac{e\hbar}{2Mc} \rho\, s_\mu \,. \tag{3.11b}$$

Therefore, since the ordinary density ρ has been counted as a slow variable, the essential change of the magnetic dipole density (3.11b) arises for the first observer from the change of the polarization vector s_μ. However, this vector has been shown to undergo precession relative to the Fermi–Walker transported observer [10] and the corresponding precession rate again contains the electric dipole density ${}^{(\mathrm{el})}M_\mu$ (3.11a). Thus, the result is that our first observer sees either both dipole densities as being rapid or neither, according to whether ${}^{(\mathrm{el})}M_\mu$ has to be counted as slow or rapid. However the electric dipole density (3.11a) is dependent on the pseudo-density $\tilde{\rho}$, and a glance at the result (2.27a) readily reveals the pseudo density $\tilde{\rho}$ as a rapid variable (if it is non-zero). Thus, the conclusion is that the first observer will see both dipole densities M_μ (3.11) as rapid and consequently he will expect the particle to be emitting dipole radiation of both the electric and magnetic kind.

(ii) It may appear somewhat academic to resort to an observer who, concomitantly with the particle, is subject to the trembling motion. In view of this argument, it seems more reasonable to choose another observer whose four-velocity is just the average (b_μ) of the particle motion $V_\mu \Rightarrow b_\mu$. For this choice, we deduce from the polarization tensor $S_{\mu\nu}$ (2.27d) the following expressions for the dipole densities

$$ {}^{(\mathrm{el})}M_\mu = \frac{e\hbar}{2Mc} \rho \sinh 2\kappa \big(\tilde{g}_\mu \sin(\chi + \zeta) - \tilde{f}_\mu \cos(\chi + \zeta)\big) \tag{3.12a}$$

$$ {}^{(\mathrm{m})}M_\mu = \frac{e\hbar}{2Mc} \rho \big(z \tilde{g}_\mu + \cosh 2\kappa\, \bar{l}_\mu\big)\,. \tag{3.12b}$$

In contrast to the preceeding result (3.11), the magnetic dipole density (3.12*b*) is now an unambiguously slow object, because it is completely free of the rapid phase χ. However, the electric density (3.12*a*) appears rapidly oscillating for our second observer and consequently he will predict purely electric dipole radiation, which will be Doppler-shifted for a laboratory observer according to the average motion of the radiating particle. Observe also, that the electric dipole density (3.12*a*) is closely related to the intrinsic velocity κ and therefore ${}^{(\mathrm{el})}M_\mu$ vanishes when this internal degree of freedom is not excited ($\kappa \to 0$); however the magnetic part (3.12*b*) is still present in this limit! This is the reason why the magnetic dipole properties of a Dirac particle are more relevant than their electric counterparts.

3.4. Exclusion of a closed universe

Up to now, we have delt with a fixed geometric background, either flat or curved. Now we include the geometry into the dynamics by applying Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} R\, G_{\mu\nu} = 8\pi \frac{L_{\mathrm{P}}^2}{\hbar c}\, T_{\mu\nu} \tag{3.13}$$

where L_{P} is Planck's length and the energy-momentum density $T_{\mu\nu}$ is assumed to be exclusively due to Dirac's wave field Ψ. Here we want to restrict ourselves to a Friedman–Robertson–Walker (FRW) universe which is usually considered to be a good model for the very early universe whose energy-momentum content was dominated by some quantum field, rather than by ordinary matter [11]. That quantum field is usually taken as some scalar field whose negative pressure in the 'false' vacuum drives the primeval inflation of the universe and thus offers a possible explanation for the observed flatness of the universe ($\leadsto$ *'flatness problem'* in cosmology). However, there is also some doubt [12, 13] whether such a scalar field really provides a serious foundation for the exclusion of closed and open universes in favour of the flat case. In this ambiguous situation, we want to point out here that the use of a *spinor field* in place of a *scalar field* clarifies the question to a certain extent, because the coupled Dirac–Einstein equations do forbid a *closed* universe! The reason is that for an isotropic homogeneous universe the Einstein equations (3.13) enforce an energy-momentum density $T_{\mu\nu}$ of the following kind

$$T_{\mu\nu} = \mathcal{M}\, b_\mu b_\nu - \mathcal{P}\mathcal{B}_{\mu\nu} \qquad \Big(\mathcal{B}_{\mu\nu} := G_{\mu\nu} - b_\mu b_\mu\Big) \tag{3.14}$$

where the mass-energy density $\mathcal{M}$ and the pressure $\mathcal{P}$ are homogeneous; and it is just such a density $T_{\mu\nu}$ which cannot be constructed with the Dirac spinor field Ψ in a *closed* universe!

For a proof of this assertion, we have to explicitly compute the (symmetrized) energy-momentum density ${}^{(\mathrm{s})}T_{(\mu\nu)}[\Psi]$ due to the spinor field Ψ; the result is [14]

$$\begin{aligned}(\hbar c)^{-1} \cdot {}^{(\mathrm{s})}T_{(\mu\nu)} &= \tfrac{1}{4} m\, \rho\, G_{\mu\nu} + \tfrac{1}{2}{}^{(\mathrm{r})}N\big[2\big(j_\mu b_\nu + j_\nu b_\mu\big) - G_{\mu\nu}\big(b_\lambda j^\lambda\big)\big] \\ &+ \tfrac{1}{2}{}^{(\mathrm{c})}\tilde{N}\big[2\big(\tilde{j}_\mu b_\nu + \tilde{j}_\nu b_\mu\big) - G_{\mu\nu}\big(b_\lambda \tilde{j}^\lambda\big)\big] \\ &+ \big({}^{(\mathrm{r})}W \cdot \rho - {}^{(\mathrm{r})}\tilde{W} \cdot \tilde{\rho}\big)\big(4 b_\mu b_\nu - G_{\mu\nu}\big) \\ &- 8\,{}^{(\mathrm{c})}W\big[b_\mu\big(b^\lambda S_{\lambda\nu}\big) + b_\nu\big(b^\lambda S_{\lambda\mu}\big)\big] \\ &- 8\,{}^{(\mathrm{c})}\tilde{W}\big[b_\mu\big(b^\lambda\, {}^*S_{\lambda\nu}\big) + b_\nu\big(b^\lambda\, {}^*S_{\lambda\mu}\big)\big]\,. \end{aligned} \tag{3.15}$$

Here, the scalar fields ${}^{(r)}N$, ${}^{(c)}\tilde{N}$, ${}^{(r)}W$, ${}^{(c)}W$, and ${}^{(c)}\tilde{W}$ are some parameters of the cosmological form of the Hamiltonian $\mathcal{H}_\mu$ and need not be specified further in the present context; the important point is that we can substitute the present new result (2.27) for the densities j_μ, $\tilde{j}_\mu$ and $S_{\mu\nu}$ into (3.15); and this yields the presence of such anisotropic terms as $\tilde{g}_\mu b_\nu + \tilde{g}_\nu b_\mu$, $\tilde{f}_\mu b_\nu + \tilde{f}_\nu b_\mu$ or $\tilde{l}_\mu b_\nu + \tilde{l}_\nu b_\mu$ in the energy-momentum density. However, the isotropy requirement (3.14) demands the vanishing of these terms and this demand implies the following three conditions

$$\sinh 2\kappa\left[{}^{(r)}N\cdot\cos(\chi+\zeta)+2\,{}^{(c)}W\cdot\sin(\chi+\zeta)\right]=z\left({}^{(c)}\tilde{N}\cdot\cosh 2\kappa-2\,{}^{(c)}\tilde{W}\right) \tag{3.16}$$

$$\sinh 2\kappa\left[{}^{(r)}N\cdot\sin(\chi+\zeta)-2\,{}^{(c)}W\cdot\cos(\chi+\zeta)\right]=0 \tag{3.17}$$

$${}^{(c)}\tilde{N}-2\,{}^{(c)}\tilde{W}\cdot\cosh 2\kappa=0\,. \tag{3.18}$$

However, there are further restrictions upon the complex scalar fields $\tilde{N}={}^{(c)}\tilde{N}-\mathrm{i}\,{}^{(r)}\tilde{N}$ and $\tilde{W}={}^{(c)}\tilde{W}+\mathrm{i}\,{}^{(r)}\tilde{W}$, which are necessary in order to satisfy the original (covariant) Dirac equation (1.8). Among these restrictions [2], we only need the following one here

$$\tilde{N}\cdot\tilde{W}=0 \tag{3.19}$$

which says that either $\tilde{N}$ or $\tilde{W}$ (or both) must be zero. Combining this with the isotropy requirement (3.18) readily yields

$${}^{(c)}\tilde{N}={}^{(c)}\tilde{W}=0\,. \tag{3.20}$$

But with the vanishing of both ${}^{(c)}\tilde{N}$ and ${}^{(c)}\tilde{W}$ the equations (3.16) and (3.17) admit nothing else than the trivial solution

$${}^{(r)}N={}^{(c)}W\equiv 0 \tag{3.21}$$

provided the intrinsic velocity κ is different from zero (for $\kappa\equiv 0$, see below). Thus, the energy-momentum density (3.15) is cut down to the desired cosmological shape ${}^{(W)}T_{\mu\nu}$

$$\begin{aligned}(\hbar c)^{-1}\,{}^{(W)}T_{(\mu\nu)}&=\tfrac{1}{4}\,m\,\rho\,G_{\mu\nu}+\left({}^{(r)}W\cdot\rho-{}^{(r)}\tilde{W}\cdot\tilde{\rho}\right)(4b_\mu b_\nu-G_{\mu\nu})\\&=3\left[\rho\left({}^{(r)}W+\tfrac{1}{12}m\right)-\tilde{\rho}\cdot{}^{(r)}\tilde{W}\right]b_\mu b_\nu-\left[\rho\left({}^{(r)}W-\tfrac{1}{4}m\right)-\tilde{\rho}\cdot{}^{(r)}\tilde{W}\right]\mathcal{B}_{\mu\nu}\end{aligned} \tag{3.22}$$

from which the energy density $\mathcal{M}$ and the pressure $\mathcal{P}$ may be read off as (cf (3.14))

$$\mathcal{M}=3\hbar c\left[\rho\left({}^{(r)}W+\tfrac{1}{12}m\right)-\tilde{\rho}\cdot{}^{(r)}\tilde{W}\right] \tag{3.23a}$$

$$\mathcal{P}=\rho\left({}^{(r)}W-\tfrac{1}{4}m\right)-\tilde{\rho}\cdot{}^{(r)}\tilde{W}\,. \tag{3.23b}$$

Evidently, both scalars $\mathcal{P}$ and $\mathcal{M}$ are dependent on the density ρ and therefore they can be homogeneous only if ρ itself is a homogeneous scalar field, i.e.

$$\partial_\mu\rho\overset{!}{\equiv}b_\mu\left(b^\nu\partial_\nu\rho\right)\,. \tag{3.24}$$

However, with the conditions (3.20), (3.21) the gradient field of the density ρ is found as [2]

$$\partial_\mu\rho=3\,{}^{(c)}N\rho\,b_\mu+4\,{}^{(r)}\tilde{N}\,b^\lambda\,{}^*S_{\mu\lambda}-2\,{}^{(r)}\tilde{W}\left(4b_\mu b_\lambda-G_{\mu\lambda}\right)\tilde{j}^\lambda \tag{3.25}$$

and consequently the homogenity requirement (3.24) demands

$$^{(\mathrm{r})}\tilde{N} = {}^{(\mathrm{r})}\tilde{W} = 0\,. \tag{3.26}$$

This result cuts the energy-momentum density down to its final form for a FRW-universe

$$(\hbar c)^{-1\,(\mathrm{W})}T_{(\mu\nu)} = 3\rho\left({}^{(\mathrm{r})}W + \tfrac{1}{12}m\right)b_\mu b_\nu - \rho\left({}^{(\mathrm{r})}W - \tfrac{1}{4}m\right)\mathcal{B}_{\mu\nu} \tag{3.27}$$

which has already been used in some previous papers [2, 10, 14].

After the homogenity and isotropy requirements have left only two non-vanishing scalar fields $^{(\mathrm{c})}N$ and $^{(\mathrm{r})}W$, one can easily complete the desired proof by applying the constraint [2]

$$(N+H)^2 = \frac{\sigma}{\mathcal{R}^2} - 4\left(W - \frac{m}{4}\right)^2 - 4\tilde{W}^2 + \tilde{N}^2 \tag{3.28}$$

which evidently *excludes the closed universe* ($\sigma = -1$)!

As mentioned above, this exclusion is essentially based upon the assumption of a non-vanishing intrinsic velocity κ. However, it has already been shown [14] that for vanishing κ, when the trembling motion is absent, the *open* universe ($\sigma = +1$) must also be excluded and one is left with the somewhat trivial situation of a flat universe ($\sigma = 0$). Therefore it seems physically reasonable to admit the trembling motion and we will then encounter a much more interesting universe [15].

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